Some recent works on conformally invariant fully nonlinear elliptic equations

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- Part I. σ_k-Nirenberg Problem
- Part II. σ_k-Loewner-Nirenberg problem
- Part III. Fully nonlinear equations invariant under Möbius transformations in two dimension

• The Nirenberg problem

Which function K on the standard 2-sphere (S^2, g) is the Gauss curvature of a metric conformally equivalent g?

• PDE: $(g_u = e^{u/2} g)$

$$-\Delta_g u + 2 = 2Ke^u, \quad \text{on } S^2.$$

•On (S^n, g), $n \geq 3$, "scalar curvature" instead of "Gauss curvature".

•PDE: $(g_u = u^{\frac{4}{n-2}} g)$

$$-\Delta_g u + \frac{(n-2)n}{4} u = \frac{n-2}{4(n-1)} K u^{\frac{n+2}{n-2}}, \quad u > 0, \quad \text{on } S^n.$$

• Necessary condition : $K > 0$ somewhere.

• A crucial ingredient: Analysis of blow up solutions

• One point blow up: $n = 2$, Alice Chang and Paul Yang; $n = 3$, Bahri and Coron, Schoen and D. Zhang; $n \geq 4$, L., under flatness order $\beta \in (n-2, n)$.

• More than one point blow up occurs in dimension $n \geq 4$: L.

• Infinite energy blow up occurs in dimension $n \geq 7$: C.C. Chen and C.S. Lin

- σ_k-Nirenberg Problem
- "σ_k-curvature" instead of "scalar curvature".
- Schouten tensor: ((M, g) Riemannian manifold)

$$A_g = (n-2)^{-1}(Ric_g - [2(n-1)]^{-1} R_g g),$$

Let

$$\lambda(A_g) = (\lambda_1, \cdots, \lambda_n) = \text{eigenvalues of } A_g.$$

Then

$$\lambda_1(A_g) + \cdots + \lambda_n(A_g) = R_g.$$

- σ_k-curvature: $\sigma_k(A_g) = \sum_{1 \le i_1 < \cdots < i_k \le n} \lambda_{i_1} \cdots \lambda_{i_k}$
- PDE:

$$\sigma_k(A_{g_u}) = K, \quad \text{on } S^n.$$

≈ 2000

- Many works on such type equations: Viaclovsky, Chang-Gursky-Yang,......

Let

$$\mathcal{A} = \{K \in C^2(S^4) \mid \boxed{K > 0, |\nabla K| + |\Delta K| > 0 \text{ on } S^4}\}.$$

Fact: There exists a (unique) continuous integer-valued function (so locally constant)

$\nabla K = 0$

$$\text{Index} : \mathcal{A} \to Z$$

satisfying, for any $K \in \mathcal{A}$ having only isolated critical points,

$$\text{Index}(K) = -1 + \sum_{\bar{x} \in S^4, \nabla K(\bar{x}) = 0, \Delta K(\bar{x}) < 0} \underbrace{\text{index}_{\nabla K}}_{(-1)^{i(\bar{x})}}(\bar{x}).$$

• Theorem A (Alice Chang, Zheng-Chao Han, Paul Yang, 2011):
For any $K \in \mathcal{A}$ satisfying $\text{Index}(K) \neq 0$, there is at least one $C^3(S^4)$ solution to

$\nabla^2 K(\bar{x})$

$$\sigma_2(A_{g_u}) = K, \quad \text{on } S^4.$$

For $n \geq 3$, Let

$$\mathcal{A} = \{K \in C^2(S^n) \mid \boxed{K > 0, |\nabla K| + |\Delta K| > 0 \text{ on } S^n\}}.$$

Fact: There exists a (unique) continuous integer-valued function (so locally constant)

$$\text{Index} : \mathcal{A} \to Z$$

satisfying, for any $K \in \mathcal{A}$ having only isolated critical points,

$$\text{Index}(K) = -1 + (-1)^n \sum_{\bar{x} \in S^n, \nabla K(\bar{x}) = 0, \Delta K(\bar{x}) < 0} \text{index}_{\nabla K}(\bar{x}).$$

• Theorem 1 (L., Luc Nguyen, Bo Wang, in preparation):
Let $n \geq 3$, $\frac{n}{2} \leq k \leq n$. Then for any $K \in \mathcal{A}$ satisfying Index$(K) \neq 0$, there is at least one $C^3(S^n)$ solution to

$$\sigma_k\left(A_{g_u}\right) = K, \quad \text{on } S^n.$$

Moreover, the total degree of all solutions is equal to Index(K).

$$\sup_{S^n} u \leq C$$

- A crucial ingredient: Analysis of blow up solutions
- For simplicity, assume $k = \frac{n}{2}$.
- Let $\{u_i\}$ be a sequence of solutions with

$$u_i(P_i) = \max_{S^n} u_i \to \infty.$$

- By L. and Nguyen, 2014 JFA,

$$u_i(x) \le C dist(x, P_i)^{\frac{2-n}{2}}, \quad x \in S^n \setminus \{P_i\},$$

$$\lim_{i\to\infty} \int_{S^n \setminus B_\delta(P_i)} u_i^{\frac{2n}{n-2}} = 0, \quad \forall\ \delta > 0.$$

$$\max_{S^n \setminus B_\delta(P_i)} u_i \le C(\delta) \min_{S^n \setminus B_\delta(P_i)} u_i, \quad \forall\ r > 0,$$

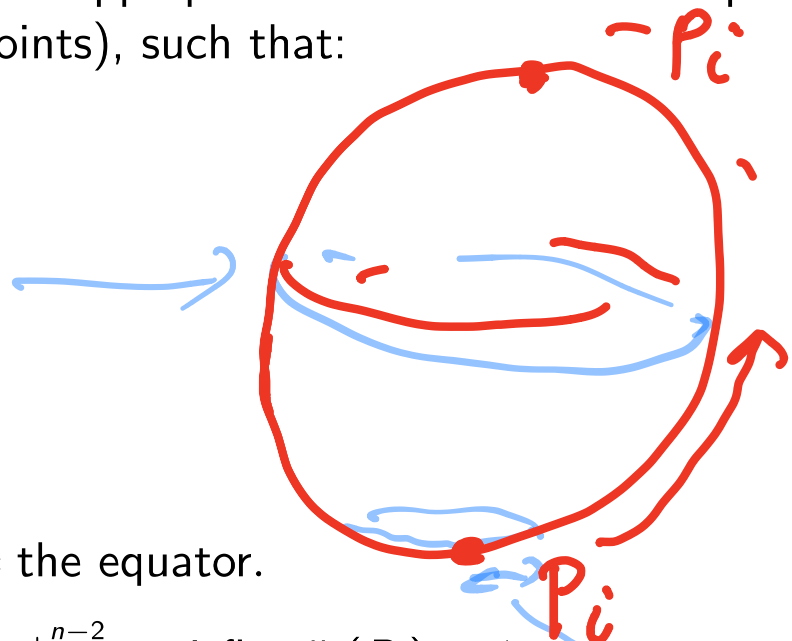
and for some $\delta_i \to 0^+$,

$$\min_{S^n} u_i \le u_i(P_i)^{-1+\delta_i}.$$

- We establish:

$$\min_{S^n} u_i \le C u_i(P_i)^{-1}.$$

• Let $\varphi_i : S^n \to S^n$ be an appropriate conformal diffeomorphism, (having $\pm P_i$ as fixed points), such that:



$\varphi_i(\partial B_{u_i(P_i)^{-\frac{2}{n-2}}}(P_i))$ = the equator.

• $\tilde{u}_i := (u_i \circ \varphi_i)|\det d\varphi_i|^{\frac{n-2}{2n}}$ satisfies $\tilde{u}_i(P_i) = 1$ ($1 = u_i(P_i)|\det d\varphi_i(P_i)|^{\frac{n-2}{2n}}$).

$u_i(P_i)^{-\frac{2}{n-2}}$

So $\tilde{u}_i(-P_i) \sim u_i(-P_i)|\det d\varphi_i(-P_i)|^{\frac{n-2}{2n}} = u_i(-P_i)u_i(P_i)$.

$K \circ \varphi_i$

• We establish: For some $\delta > 0$, $\sup\limits_{B_\delta(-P_i)} \tilde{u}_i \leq C(\delta)$.

$A^u = w\nabla^2 w - \frac{1}{2}|\nabla w|^2 I,$

$w = u^{-\frac{2}{n-2}}$

- Two proofs for this.
- First proof makes use of the following

Proposition 1. Let $B_1 \subset R^n$, $n \geq 3$, $0 < u \in C_{loc}(R^n \setminus B_1)$ satisfies

$$\lambda(A^u) \text{ is not in } \overline{\Gamma}_{n/2} \text{ or } \sigma_{n/2}(A^u) < 1, \quad \text{in } R^n \setminus B_1,$$

substitute

and

$$\limsup_{|x|\to\infty} |x|^{n-2} u(x) < \infty.$$

Assume for $C_0 > 0$, $\alpha \in (0,1]$,

$$u(x) \leq C_0 |x|^{\frac{2-n}{2}(1+\alpha)} \quad \text{in } R^n \setminus B_1,$$

then

$$u(x) \leq C_1(n, C_0, \alpha)|x|^{2-n} \quad \text{in } R^n \setminus B_1.$$

Second Proof:

Proposition 2. Let $n \geq 3$ even, $k = n/2$, there exists $\delta = \delta(n) > 0$ such that if $0 < u \in C^2(B_2)$ satisfies

$$\sigma_k(\lambda(A^u)) \leq 1, \ \lambda(A^u) \in \Gamma_k, \quad \text{in } B_2,$$

small energy

and

$$\int_{B_2} u^{\frac{2n}{n-2}} < \delta.$$

Then

$$u \leq C(n) \ \text{ in } B_1.$$

$\to L^\infty$

• For $n = 4$, $k = 2$, the above was proved by Zheng-Chao Han 2004.

• For $k < \frac{n}{2}$, a stronger version in a punctured ball was proved by Maria del Mar Gonzalez 2006.

Part II σ_k-Loewner-Nirenberg problem

$g_u = u^{\frac{4}{n-2}} g_{flat}$

- The Loewner-Nirenberg problem

Theorem B. (Loewner, Nirenberg): Let $\Omega \subset R^n$ be bounded smooth open set, $n \geq 3$. There exists a unique smooth positive solution to

$$\Delta u = u^{\frac{n+2}{n-2}}, \text{ in } \Omega,$$

$$\underline{u(x) \to \infty \text{ as } x \to \partial\Omega.}$$

$R_{g_u} = -1$

Moreover

$$\lim_{x \to \partial\Omega} \underline{dist(x, \partial\Omega)^{\frac{n-2}{2}} u(x) = c(n)} > 0.$$

σ_k

Theorem 2. (Gonzalez, L., Nguyen, 2018): Let $\Omega \subset R^n$ be bounded smooth open set, $n \geq 3$, $2 \leq k \leq n$. There exists a unique positive viscosity solution to

$$\sigma_k(-A^u) = 1, \text{ in } \Omega,$$

$$u(x) \to \infty \text{ as } x \to \partial\Omega.$$

σ_1

$\updownarrow$

$\Delta u = u^{\frac{n+2}{n-2}}$

Moreover $u \in C^{0,1}_{loc}(\Omega)$, and

$$\lim_{x \to \partial\Omega} dist(x, \partial\Omega)^{\frac{n-2}{2}} u(x) = c(n,k) > 0.$$

• Chang, Han, Yang, 2005 proved that the problem has no radially symmetric C^2 solution on any annulus.

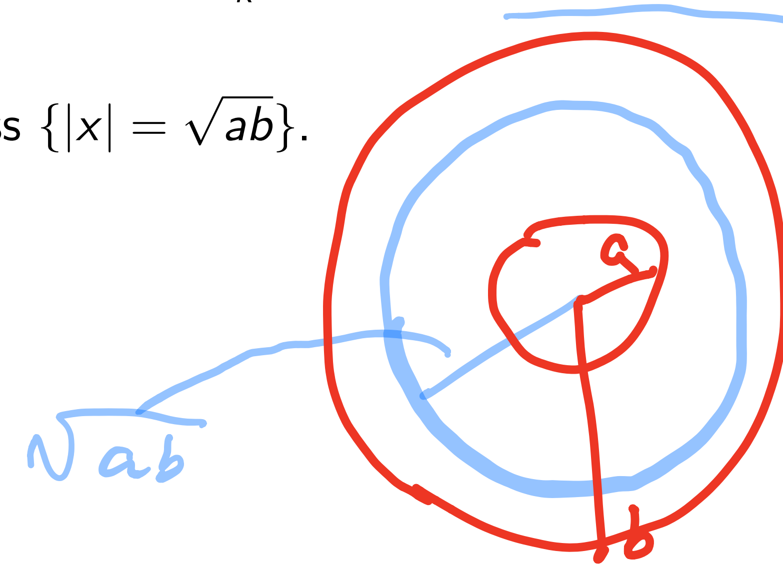
• A combination of the two results imply that there is no C^2 solution on any annulus.

Theorem 3. (L. and Nguyen, 2020) Let $\Omega = \{a < |x| < b\}$ be an annulus, $n \geq 3, 2 \leq k \leq n$. Then the solution of the σ_k-Loewner-Nirenberg problem is radially symmetric,

(i) u is C^∞ in each of $\{a < |x| < \sqrt{ab}\}$ and $\{\sqrt{ab} < |x| < b\}$,

(ii) u is $C^{1,\frac{1}{k}}$ but not $C^{1,\gamma}$ with $\gamma > \frac{1}{k}$ in each of $\{a < |x| < \sqrt{ab}\}$ and $\{\sqrt{ab} < |x| < b\}$,

(iii) and $\partial_r u$ jumps across $\{|x| = \sqrt{ab}\}$.



Theorem 4. (L. and Nguyen, 2020) Let $\Omega \subset R^n$ be bounded open, $n \geq 3$. Then there is no positive $u \in C^2(\Omega)$ such that $\lambda(-A^u) \in \overline{\Gamma}_2$ in Ω and that $(\Omega, u^{\frac{4}{n-2}} g_{flat})$ admits a smooth minimal immersion $f : \Sigma^{n-1} \to \Omega$ for some smooth compact manifold Σ^{n-1}

Subsolut

Corollary Let Ω be an annulus in R^n, $n \geq 3$ Then there is no radially symmetric positive $u \in C^2(\Omega)$ such that $\lambda(-A^u) \in \overline{\Gamma}_2$ in Ω and $u(x) \to \infty$ as $x \to \partial\Omega$.

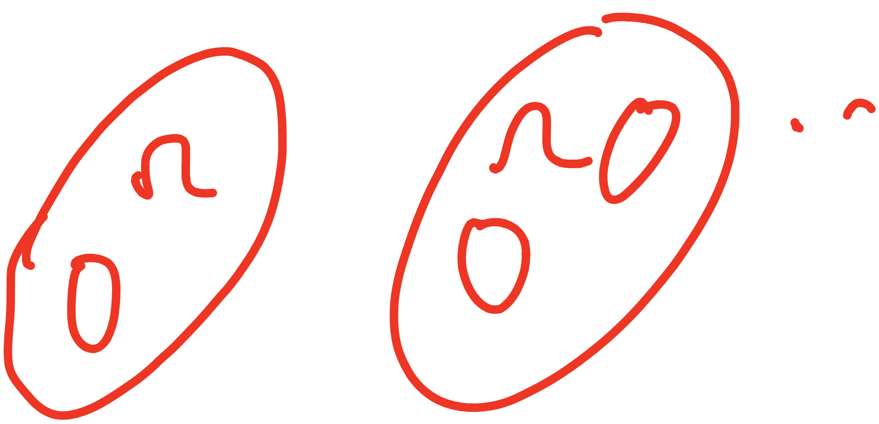
The proof is based on

Lemma Let $\Omega \subset R^n$ be open, $n \geq 3$, $\Sigma^{n-1} \subset \Omega$ smooth. If $u \in C^2(\Omega)$ satisfies $\lambda(-A^u) \in \overline{\Gamma}_2$ in Ω, then

$$\Delta_\Sigma u + \frac{n-2}{4(n-1)} |H_{\Sigma,u}|^2 u^{\frac{n+2}{n-2}} - \frac{n-2}{4(n-1)} |H_\Sigma|^2 u - \frac{1}{(n-2)u} |\nabla_\Sigma u|^2 \geq 0 \text{ on } \Sigma.$$

mean curvature of Σ,
w.r.t. $g_u = u^{\frac{4}{n-2}} g_{flat}$

Theorem 5 (L., Luc Nguyen, Jingang Xiong, in preparation). Let $\Omega \subset R^n$ be bounded, connected, smooth open, $n \geq 3$. Assume $\partial\Omega$ has more than one connected component. Then, for any $2 \leq k \leq n$, the σ_k-Loewner-Nirenberg problem has no $C^2(\Omega)$ solution.



Question : If $\Omega \subset \mathbb{R}^n$ strictly convex, bounded
$\partial\Omega$ smooth
is the solution to the σ_k - Loewner - Nirenberg problem always C^2 ?

Part III Fully nonlinear equations invariant under Möbius transformations in two dimension

- Define

$$A^u := -e^{-u}\nabla^2 u + \frac{1}{2}e^{-u} du \otimes du - \frac{1}{4}e^{-u}|\nabla u|^2 I.$$

For a function u, and for a Möbius transformation ψ, denote:

$$u_\psi := u \circ \psi + \ln |J_\psi|.$$

Jacobian of ψ

- $H \in C^0(R^2 \times R \times R^2 \times \mathcal{S}^{2\times 2})$ is invariant under Möbius transformations, i.e.,

$$H(\cdot, u_\psi, \nabla u_\psi, \nabla^2 u_\psi) = H(\cdot, u, \nabla u, \nabla^2 u) \circ \psi \text{ for all } u \text{ and } \psi,$$

if and only if

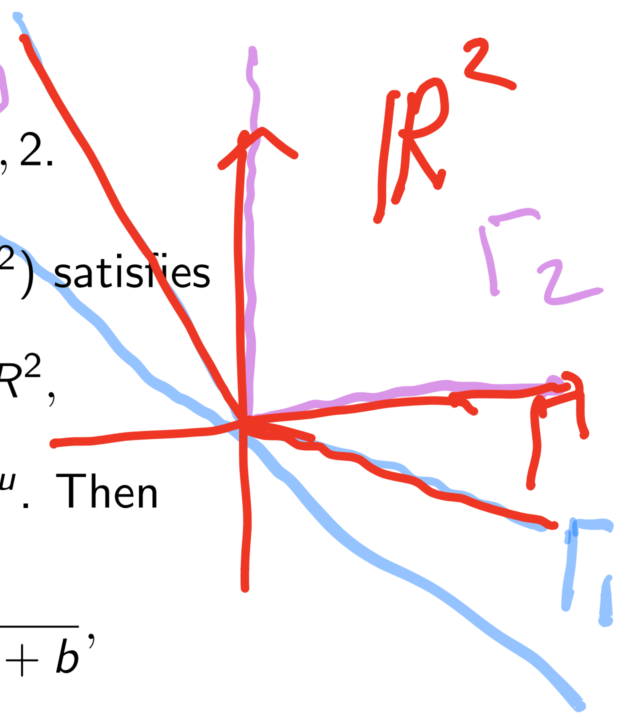
$$H(\cdot, u, \nabla u, \nabla^2 u) \equiv F(A^u)$$

for some $F \in C^0(\mathcal{S}^{2\times 2})$ which is invariant under orthogonal conjugation.

$F(M) = f(\lambda(M))$

Let Γ be open convex symmetric cone in R^2 with vertex at the origin satisfying $\Gamma_2 \subset \Gamma \subsetneq \Gamma_1$, where
$\Gamma_1 := \{\lambda_1 + \lambda_2 > 0\}$, $\Gamma_2 = \{\lambda_1, \lambda_2 > 0\}$.
Let $f \in C^1(\Gamma)$ satisfy $\partial_{\lambda_i} f > 0$ in Γ, $i = 1, 2$.
Theorem 6 (L., Han Lu, Siyuan Lu).
Let (f, Γ) be as above. Assume $u \in C^2(R^2)$ satisfies

$$f(\lambda(A^u)) = 1, \quad \text{in } R^2,$$



where $\lambda(A^u)$ denotes the eigenvalues of A^u. Then

$$\Longrightarrow \qquad u(x) = 2 \ln \frac{8a}{8|x - x_0|^2 + b},$$

for some $x_0 \in R^2$ and some positive constants a and b satisfying $(b/(2a^2), b/(2a^2)) \in \Gamma$ and $f(b/(2a^2), b/(2a^2)) = 1$.

No assumption on u at ∞ needed

- C. Li and W. Chen 1991 proved: Let $u \in C^2(R^2)$ satisfy

$$-\Delta u = e^u, \quad \text{in } R^2,$$

and

$$\int_{R^2} e^u < \infty.$$

Then

$$u(x) = 2\ln \frac{8a}{8|x - x_0|^2 + a^2},$$

for some $x_0 \in R^2$ and some positive constant a.

- This corresponds to $\Gamma = \Gamma_1$ and $f(\lambda_1, \lambda_2) = \lambda_1 + \lambda_2$.
- In this case, condition $\int_{R^2} e^u < \infty$ can not be dropped.

THANK YOU!