Several Questions Related to Homogenization

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June 16, 2020

Outline Of The Talk

Notions of Weak Convergences and Strong Estimates

Two Problems: Inverse and Boundary Control

Liouville Type Theorems, Elliptic Estimates and Ancient Solutions

Asymptotics of Eigenvalues and Eigenfunctions

Further Remarks

G and H Convergences

Let $L_i u \equiv \partial_{x_k}\left(a^i_{k,j}(x)\frac{\partial u}{\partial x_j}\right)$, and such that

$$\lambda|\xi|^2 \le a^i_{k,j}(x)\xi_k\xi_j \le \Lambda|\xi|^2, \quad \text{for} \quad \xi \in \mathbb{R}^n, \quad i = 1, 2, \cdots .$$

Consider

$$\begin{cases} L_i u_i = f & \text{in} \quad \Omega, \\ u_i = g & \text{on} \quad \partial\Omega, \end{cases}$$

here Ω is a bounded (Lipschitz) domain, $f \in H^{-1}(\Omega)$ and $g \in H^{\frac{1}{2}}(\partial\Omega)$. Then there is a subsequence $L_{i'} \xrightarrow{G} L$, where L is an operator of the same form as L_i 's :

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that is, $u_i \rightharpoonup u$ in $H^1(\Omega)$ with $u\,|_{\partial\Omega} = g$ and $Lu = f \in H^{-1}(\Omega)$.

(De Giorgi and Spagnolo, Tartar-Murat).

Note. From Div-Curl lemma, one has

$$\int_\Omega a^i_{k,j}(x) u^i_{x_k} u^i_{x_j} dx \to \int_\Omega a_{k,j}(x) u_{x_k} u_{x_j} dx.$$

Operators with Rapidly Oscillating Periodic Coefficients

$$L_\varepsilon = \frac{\partial}{\partial x_i}\left[a_{ij}^{\alpha\beta}\left(\frac{x}{\varepsilon}\right)\frac{\partial}{\partial x_j}\right], \quad \varepsilon > 0.$$

Here

$$A = A(y) = \left(a_{ij}^{\alpha\beta}(y)\right), \;\; 1 \le i, j \le n, 1 \le \alpha, \beta \le m$$

satisfies

• A is real and uniformly elliptic.

• A has some smoothness (e.g., A is Hölder or Lipschitz continuous).

• A is periodic w. r. t. $\mathbb{Z}^n$: $A(y+z) = A(y)$ for all $y \in \mathbb{R}^n, \quad z \in \mathbb{Z}^n$.

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Homogenization

For the boundary value problem

$$\begin{cases} L_\varepsilon(u_\varepsilon) = \operatorname{div} F \quad \text{in} \quad \Omega \\ u_\varepsilon \quad \text{subject to certain boundary condition,} \end{cases}$$

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One has, as $\varepsilon \to 0$,

$u_\varepsilon \to u_0$ strongly in $L^2(\Omega)$ and weakly in $H^1(\Omega)$, where u_0 is a solution of an elliptic system with constant coefficients:

$$\begin{cases} L_0(u_0) = \text{div } F \quad \text{in} \quad \Omega \\ u_0 \quad \text{subject to the same boundary condition.} \end{cases}$$

Here $L_0 = \text{div}(\hat{A}\nabla)$ and $\hat{A} = \left(\hat{a}_{ij}^{\alpha\beta}\right)$ may be computed "explicitly" using $A(y)$.

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In mid-1980s, Avellaneda and I became interested in a Quantitative Theory of Homogenization partially due the following two problems.

An Inverse problem [J.L. Lions]

$$(*) \begin{cases} \frac{\partial u_\varepsilon}{\partial t} = \frac{\partial}{\partial x_i}\left(a^{ij}\left(\frac{x}{\varepsilon}\right)\frac{\partial u_\varepsilon}{\partial x_j}\right), & \text{in } \Omega \times (0, T_0), \\ u_\varepsilon(x, 0) = \phi(x), & x \in \Omega, \\ u_\varepsilon(x, t) = 0, & x \in \partial\Omega \times [0, T_0]. \end{cases}$$

Problem. One observes at time $t = T_0$ a possible solution of u_ε of $(*)$ to find a function $f_\varepsilon(x)$. How can one construct a solution of $(*)$ for $0 < t < T_0$? Here $\varepsilon \le \delta \le\le 1$.

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Question a. How can one assert that $f_\varepsilon(x)$ is actually close to $u_\varepsilon(x, T_0)$, for a solution of $(*)$? Is there a criteria?

Question b. If $f_\varepsilon(x)$ is indeed close to some $u_\varepsilon(x, T_0)$, then is it possible to construct $u_\varepsilon(x, t)$ for $0 < t < T_0$?

Question b above means: if $\|f_\varepsilon(x) - u_\varepsilon(x, T_0)\| \leq \delta$, can one construct from $f_\varepsilon(x)$ an approximate solution $V^\delta(x, t)$ such that

$$\|V^\delta(x, t) - u_\varepsilon(x, t)\| \leq O(\delta^\alpha) \text{ for } 0 < t_0 \leq t \leq T_0\,?$$

Here α is a positive number.

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Well Posed and Well Behaved Problems

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a) Existence, b) Uniqueness,
c) Stability (Continuous dependence).

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For example, in periodic homogenizations:

$\varepsilon =$period size, $\quad\quad \Delta =$ grid size,
$u_\varepsilon =$ solution to the ε-problem,
$u_\Delta =$ solution to the Δ-cell problem (Numerical),
$u_0 =$solution to the homogenized problem.

Question. How close u_Δ to u_ε?

Total error bound $\leq O(\Delta^\beta) + O\left(\omega\left(\frac{\varepsilon}{\Delta}\right)\right)$
$\omega(\delta) =$theoretical error bound of $\|u_\delta - u_0\|$.
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Thus one needs $\omega(\delta) \leq O(\delta^a)$ for as large a as possible.

But in some stochastic homogenizations or fully nonlinear elliptic homogenizations, the best known $\omega(\delta)$ may be given by

$$|\log \delta|^{-a}, \quad \text{for} \quad \text{some} \quad \text{small} \quad a > 0.$$

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Solution [L-, In a Special Osher's Volume 2003]

$$\begin{cases} \frac{\partial u^\varepsilon}{\partial t} - L_\varepsilon u^\varepsilon = 0 & \text{in } \Omega \times (0, T) \\ u^\varepsilon(x,t) = 0 & \text{on } \partial\Omega \times (0, T) \end{cases}$$

$u^\varepsilon(x,0)$ unknown, but $\|u^\varepsilon(x,0)\|_{L^2(\Omega)} \leq 1$.

(i) [TEST] If the observed data $f^\varepsilon(x)$ at $t = T$ to be close to $u^\varepsilon(x,T)$ within a δ error, then

$$|\int_\Omega f^\varepsilon(x)\phi_k(x)dx| \leq e^{-\lambda_k T} + \delta.$$

Here $k = 1, 2, \cdots, N_0$.
How large N_0 needs to be for this test?

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Suppose $T = 1$ and $|\Omega| = 1$, then Weyl's asymptotic formula yields,

$$\lambda_k = c(d)\left(\frac{k}{|\Omega|}\right)^{\frac{2}{d}} = c(d)k^{\frac{2}{d}}.$$

Here $d = \dim\ \Omega$ ($d = 1, 2, 3$ for examples).

If $N_0 = 10$, then $\lambda_{N_0} = \frac{10^2}{\pi^2}$ $(d = 1) \geq 10$.

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A construction

(ii) Let

$$\widetilde{u}^{\varepsilon}(x,t) = \sum_{k=1}^{N_0} c_k e^{-\lambda_k (T-t)} \phi_k(x),$$

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$$|\widetilde{u}^{\varepsilon}(x,t) - u^{\varepsilon}(x,t)| \leq C(\Omega)\delta^{a}$$

for all $0 < t_0 \leq t \leq T$. Here $a = a(t_0) > 0$, and $\varepsilon \leq \delta$.

Where $\{(\lambda_k, \phi_k), k = 1, \cdots, N_0\}$ are eigenvalues and normalized eigenfunctions of the homogenized operator for L_ε.

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Boundary Controllability in Oscillating Medium

$$(**)\begin{cases} \Box\phi = \phi_{tt} - \phi_{xx} = 0 & \text{in } \Omega \times (0, T), \\ \phi(0) = \phi^0, \quad \phi'(0) = \phi^1 & \text{on } \Omega, \\ \phi = g & \text{on } \partial\Omega \times (0, T) \end{cases}$$

Question: Can one find a function g supported on $\partial\Omega \times (0, T)$ such that $\phi(T) = \phi_t(T) = 0$.

The smallest T that works is called the Optimal control time, and that g with the smallest L^2 norm is called an Optimal control.

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Here y is the solution of $(**)$ with zero Dirichlet boundary condition and initial (y_0, y_1).

This defines a map

$$\Lambda\{y_0, y_1\} = \{\psi'(0), -\psi(0)\}.$$

Suppose Λ is invertible in an appropriate Hilbert space (for T suitably large), then the problem $(**)$ can be solved. Indeed, given ϕ_0, ϕ_1, we solve $\Lambda\{y_0, y_1\} = \{-\phi_1, \phi_0\}$. Then we choose $g = \psi|_{\partial\Omega\times(0,T)}$, one solves the control problem.

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To see Λ is invertible, via Lax-Milgrame, one needs to show the so-called coerciveness:

$$(\Lambda\{y_0, y_1\}, \{y_0, y_1\}) = \int_{\Gamma\times(0,T)} \left|\frac{\partial y}{\partial \nu}\right|^2 d\Gamma dt.$$

If T is large enough, then one has the Observability Inequality

$$\int_{\Gamma\times(0,T)} \left|\frac{\partial y}{\partial \nu}\right|^2 d\Gamma dt \geq c_0 \|\{y_0, y_1\}\|^2$$

with norm taken on $H_0^1(\Omega) \times L^2(\Omega)$.
$\Lambda : \quad H_0^1(\Omega) \times L^2(\Omega) \quad \leftrightarrow \quad H^{-1}(\Omega) \times L^2(\Omega)$ is an isomorphism.

J.L.Lions posed the same exact boundary control problem for waves in an oscillating Medium, that is, with the Laplacian operator being replaced by L_ε. Z. Shen and I have recently obtained a partial result. Our result says that such boundary controllability is valid if one is willing to restricted it to the first $O(\varepsilon^{-d/3})$ numbers of Fourier modes, $d > 1$.
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A related problem to the Boundary Controllability

Let

$$L_\varepsilon = \partial_{x_i}\left[a^{ij}(x/\varepsilon)\partial_{x_j}\right], \quad \text{and} \quad \int_\Omega u_\varepsilon^2 = 1$$

be such that $L_\varepsilon u_\varepsilon + \lambda_\varepsilon u_\varepsilon = 0$ in Ω.

Question. Is it true that

$$c\lambda_\varepsilon \leq \int_{\partial\Omega} \left|\frac{\partial u_\varepsilon}{\partial \nu_\varepsilon}\right|^2 d\sigma \leq C\lambda_\varepsilon ?$$

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Global structure of solutions and Liouville type theorems.

$$\frac{dx}{dt} = A(t)x$$

with $A(t)$ periodic in t.

The fundamental solution matrix

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Let L be an uniformly elliptic operator with periodic coefficients in $\mathbb{R}^n$. Suppose L is either of the form $L = \text{div } (A(x)\nabla)$
or of the form $L = -a^{ij}(x)\partial^2_{x_i,x_j}$. Then

Theorem. (Avellaneda - L, 1989).
If $Lu = 0$ in $\mathbb{R}^n$, and $\max_{B_R} |u(x)| \leq MR^m$, for a sequence of $R \to \infty$, then U is a polynomial of degree $\leq m$ with periodic coefficents.

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Very recently, Armstrong-Kuusi-Smart proved a large scale analyticity of solutions. It is a striking result and it has many implications. In particular, entire solutions of growth bounded by $O(Exp(cR))$, for a small positive c, can be written as a locally uniformly and absolutely convergent series of L harmonic polynomials. On the other hand, one can show also that, for any harmonic power series with a small exponential growth, there is a $L-$ harmonic power series that converges locally uniformly on the whole space.

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Remark. (a) Moser-Struwe (1992) proved that there are solutions of the PDE:

$$-\Delta u + v(x, u) = 0 \qquad \text{in} \quad \mathbb{R}^n,$$

with $u(x) = \alpha \cdot x + B_\alpha(x)$, for any $\alpha \in \mathbb{R}^n$, and $B_\alpha(x)$'s are bounded.

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Quantitative Theory

Theorem. (M. Avellaneda and F. Lin, 1991)

Let G and G_0 be Green Functions of L and L_0, then

$$|G(x,y) - G_0(x,y)| \leq \frac{c_1}{|x-y|^{n-1}}$$

$$|\nabla_x G(x,y) - P(x)\nabla_x G_0(x,y)| \leq \frac{c_2}{|x-y|^{n}}$$

$$|\nabla_x \nabla_y G(x,y) - P(x)P(y)\nabla_x \nabla_y G_0(x,y)| \leq \frac{c_3}{|x-y|^{n+1}},$$

for some positive constants c_1, c_2, c_3 and periodic matrix $P(x)$,

and for all $(x,y) \in \mathbb{R}^n \times \mathbb{R}^n$ with $|x-y| \geq 1$.

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Consequences:

(i) Operators $\frac{\partial}{\partial x^\alpha}(L)^{-1}\frac{\partial}{\partial x^\beta}$,

$$\frac{\partial}{\partial x^\alpha}(L)^{-1/2}, \qquad (L)^{-1/2}\frac{\partial}{\partial x^\beta}, \qquad 1 \leq \alpha, \qquad \beta \leq n,$$

are all bounded from $L^p(\mathbb{R}^n)$ to $L^p(\mathbb{R}^n)$, $1 < p < \infty$

and from $L^1(\mathbb{R}^n)$ into weak - $L^1(\mathbb{R}^n)$.

(ii) If $Lu = \text{div} \overrightarrow{F}$ in $\mathbb{R}^n$, then

$$\|u\|_{W^{1,p}(\mathbb{R}^n)} \leq C\| \overrightarrow{F} \|_{L^p(\mathbb{R}^n)}.$$

(iii) The operators $\frac{\partial}{\partial x^\alpha}\frac{\partial}{\partial x^\beta}(L)^{-1}$ are bounded from

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(v) If $Lu = 0$ in H^n_+ (a half-space),

$u = f$ on ∂H^n_+ ,

then $\|u^*\|_{L^p(\partial H^n_+)} \leq c\|f\|_{L^p(\partial H^n_+)}, \quad 1 < p < \infty.$

Here u^* is the usual Hardy-Littlewood maximal function.

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Over the last decade, there have been impressive new results for Quantitative Periodic and Stochastic Homogenization, for regularity and rates of convergence,. . . by many authors: Armstrong, Kuusi, Smart, Gloria, Otto, P.L.Lions, Le Bris,Kenig, Shen, Zhuge,. . . For homogenization of quasilinear elliptic equations and Hamilton-Jacobi equations, the corresponding studies were made by Lions-Papanicolaou-Varadhan, Evans, Souganidias, Armstrong, Smart. . .. And fully nonlinear differential and fully nonlinear integral equations with periodic coefficents were studied by L. Cafferelli, P.L.Lions, Souganidias and

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For the Dirichlet and Neumann Boundary value problems, there are hard upto boundary uniform estimates.

• In case of elliptic equations or systems in $C^{1,\alpha}$ domains, the L^p-Dirichlet problem for $L_\varepsilon(u_\varepsilon) = 0$ was established in [AL, 1987], and for $m = 1$ (equations) for Lipschitz domains and $2 - \delta < p < \infty$ was solved by B. Dalhberg (1990, unpublished).

• For the elliptic systems, the L^2-Dirichlet and Neumann, and the regularity problems for $L_\varepsilon(u_\varepsilon) = 0$ in Lipschitz domains were solved by Kenig-Shen (2009) using the method of layer potentials.

• Kenig-Lin-Shen (2010): L^p Neumann and regularity problems as well as representations by layer potentials were solved for $1 < p < \infty$.

Ancient solutions of the heat equation

Liouville type theorems can be extended to the parabolic case for periodic operators and beyond.

$$u_t - \Delta u = 0, \text{ in } R^n \times (-\infty, 0).$$

Theorem (L-Zhang) Let u be a nonnegative ancient solution to the heat equation in $\mathbb{R}^n \times (-\infty, 0]$. Then $u(x, -t)$ is completely monotone function in t. Moreover, there exists a family of Radon measures $\mu(\cdot, s)$ on the unit sphere $\mathbb{S}^{n-1}$, $s \in [0, \infty)$, and a Radon measure ρ on $\mathbb{R}+$ such that

$$u(x,t) = \int_0^\infty \int_{\mathbb{S}^{n-1}} e^{ts + \sqrt{s}\xi \cdot x} d\mu(\xi, s) d\rho(s).$$

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Our proof for the Euclidian space works also for positive ancient solutions on a class of complete Riemannian manifolds as well as for operators with periodic coefficients.

$$u_t - \Delta_{\mathcal{M}} u = 0 \text{ on } \mathcal{M} \times (-\infty, 0].$$

Here $\mathcal{M}$ is a complete Riemannian manifold with nonnegative Ricci curvature.

Theorem (L-Zhang) Let u be a nonnegative ancient solution of the heat equation on $\mathcal{M}^n \times (-\infty, 0]$, where $\mathcal{M}$ is a complete (noncompact) Riemann manifold with nonnegative Ricci curvature. Then there is a family of nonnegative Radon measure $\mu(\cdot, s)$ on the family of Martin-Boundaries Σ_s of (**) such that

$$u(x,t) = \int_0^\infty \int_{\Sigma_s} e^{st}\mathcal{P}_s(x,\omega)d\mu(\omega,s)d\rho(s).$$

Question Under what conditions for a complete Riemannian manifold $\mathcal{M}$ that yield all Σ_s would be the same? And is it possible to obtain an explicit expression for minimal solutions $\mathcal{P}_s(x,\omega)$? Can one link the Martin-boundaries with the geometric asymptotics of $\mathcal{M}$?

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Remarks Understanding positive solutions of (**) is closely related to the classical Choquet's Theorem in convex and functional analysis. In the work of Caffarelli-Littman for $\mathcal{M} = \mathbb{R}^n$, an explicit identification (representation) of Martin Boundary was given.

There are many other works by Widder, Karpelevic, J.C.Taylor, Murata, Pinchover, Korani, Anderson-Schoen, A.Ancona... on positive harmonic and coloric functions and on the study of the Martin boundary.

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Eigenvalue and Eigenfunction Asymptotics

Progress on the J.L.Lion's problem: For a special class of L_ε in 1-D, Avellaneda-Bardos-Raugh constructed counter-examples. Castro-Zuazua constructed finer examples for $\lambda_\varepsilon \simeq \varepsilon^{-2}$ that

$$\int_{\partial\Omega} \left| \frac{\partial u_\varepsilon}{\partial \nu_\varepsilon} \right|^2 d\sigma \sim \lambda_\varepsilon^{3/2},$$

for the upper bound, and lower bound could be exponentially small.

Castro-Zuazua showed in 1-D case that

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if either $\lambda_\varepsilon << \varepsilon^{-2}$ or $\lambda_\varepsilon >> \varepsilon^{-2}$.

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Absence of L^2 Eigenvalues and Quantitative Continuation

a) It is possible to have a compact supported smooth eigenfunction for such equations when the coefficients are only Holder continuous [N.Filonov].

b) It means also the absence of L^2 embedded eigenvalues may not be true in general for such elliptic operators with only Holder continuous coefficients.

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Convergence Rates of Eigenvalues (Kenig-Lin-Shen 2009)

$$|\lambda_\varepsilon^k - \lambda_0^k| \le O(\varepsilon)$$
$$|\mu_\varepsilon^k - \mu_0^k| \le O(\varepsilon)$$
$$|\eta_\varepsilon^k - \eta_0^k| \le O(\varepsilon)$$

for $\varepsilon \le \varepsilon_k$ if $\partial\Omega$ is $C^{1,1}$.

If Ω is Lipschitz, then

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for any $a > \frac{1}{2}$ and $\varepsilon \le \varepsilon_k$.

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